A hierarchical multi-levels energy method for controllability of coupled systems of evolution equations and applications

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Abstract. We consider exact controllability issues for controlled coupled systems of evolution equations in Hilbert spaces. The common goal in this framework is to determine whether if it is possible to steer the solution of these systems from any initial state to any final state by choosing appropriate controls.
Many applications in control theory are further concerned with the controllability of coupled systems of PDE's by a *reduced* number of controls. This situation occurs whenever the number of controls is strictly less than the number of equations. Such problems also appear when one wants to build robust controls for scalar equations, or when one wants to control simultaneously devices in parallel. We focus in this talk on performing exact controllability for coupled systems of evolution equations with less controls than usually required.
We present results for coupled cascade systems. Such systems involve matrix operators in triangular form and naturally appear in questions related to insensitizing control. We give applications to examples of cascade systems of coupled hyperbolic, parabolic and Schrödinger PDE's and to the existence of insensitizing and simultaneous controls. These results are based on a sharp analysis of abstract controlled coupled systems by means of energy methods, transposition solutions for coupled systems and some invariance properties of the solutions. In particular we generalize the two-level energy method we introduced initially for two coupled systems of evolution equations, to a hierarchical multi-levels energy method for n-coupled cascade systems of bi-diagonal or mixed bi-diagonal type.

Asymptotic stabilization of the hyperelastic-rod wave equation

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Abstract. We investigate the problem of asymptotic stabilization of the hyperelastic-rod wave equation on the real line

$$\partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma \left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) \quad t > 0,\ x \in \mathbb{R}, \tag{1}$$

where $u(t,x)$ represents the radial deformation in a cylindrical compressible hyperelastic rod, and $\gamma \in \mathbb{R}$ is some given constant depending on the material and on the prestress of the rod. Observe that if $\gamma = 1$, then (1) is the classical Camassa–Holm equation modelling the propagation of unidirectional shallow water waves on a flat bottom.
The asymptotic stabilizability of the Camassa–Holm equation through a stationary feedback law was recently established, within the space of H^2 solutions, by O. Glass (2008) by means of a forcing term acting as a control, and by V. Perrollaz (2010) by means of a boundary feedback. Here, we assume $\gamma > 0$, and consider the equation (1) with an additional force term of the form

$$f : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R}), \qquad f[u] = -\lambda(u - \partial^2_{xx} u),$$

for some $\lambda > 0$. With the same approach introduce by A. Bressan and A. Constantin (2007), we show the existence of a semigroup of global weak dissipative solutions of the corresponding closed-loop system

$$\partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma \left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) + f[u] \qquad t > 0,\ x \in \mathbb{R}, \tag{2}$$

defined for every initial data $u_0 \in H^1(\mathbb{R})$, and we prove that any such solution decays esponentially to 0 as $t \to \infty$.
Joint work with: Giuseppe M. Coclite (University of Bari)

Controllability results
for degenerate parabolic operators

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Abstract. Unlike uniformly parabolic equations, parabolic operators that degenerate on subsets of the space domain exhibit very different behaviors from the point of view of controllability. For instance, null controllability in arbitrary time may be true or false according to the degree of degeneracy and there are also examples where a positive minimal time is needed to ensure such a property. This talk will survey most of the theory that has been established so far for operators with boundary degeneracy, and discuss recent results for operators of Grushin type and of Kolmogorov type, which degenerate in the interior. Inverse problems for these operators will also be investigated.
Joint work with: Piermarco Cannarsa (Università degli Studi di Roma Tor Vergata), Roberto Guglielmi (Università degli Studi di Roma Tor Vergata.

Qualitative and optimal control properties
of a mathematical model for a certain fluid-structure interaction

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Abstract. The talk will focus on a system of Partial Differential Equations (PDE) which describe the interactions of a fluid flow in a three-dimensional bounded domain, with the transversal displacement of an elastic part of its boundary. More precisely, the mathematical model comprises a Stokes system for the fluid velocity field and a classical Kirchhoff equation for the elastic deformation of the plate; the resulting coupling is of parabolic/hyperbolic type. Our ultimate goal is to address solvability of associated optimal control problems (with quadratic functionals) on an infinite time horizon, with the PDE dynamics being under the action of point control on the elastic plate. Preliminary to this control theoretic objective however, several mathematical issues for the fluid-structure model must first be addressed. These include: (i) a semigroup well-posedness setup for the uncontrolled PDE system; this will constitute an advantageous alternative to the setup resulting from the Galerkin approximation scheme which has been invoked in the literature; (ii) verification for a requisite uniform stability property (to hold in the absence of mechanical dissipation); such is needed for proper consideration of the aforesaid optimal control problem on infinite horizon; (iii) the derivation of trace regularity results for the pressure of the fluid, corresponding to arbitrary finite energy data. (Such an additional regularity result would also be of intrinsic interest). (The talk is based on ongoing work with George Avalos (University of Nebraska).)

Domain invariance for finite and infinite
dimensional diffusion processes

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Abstract. We shall discuss the invariance of a state domain with respect to the flow associated with a given diffusion process. The finite dimensional problem will be addressed first. In this

case, we will derive necessary and sufficient conditions for invariance in terms of the distance function from the boundary of the domain. We shall then turn to infinite dimensional processes, describing the difficulties one has to cope with in order to extend the above method to solutions of stochastic partial differential equations. In particular, we shall report on recent results obtained in collaboration with Da Prato for stochastic reaction-diffusion systems and models in population dynamics.

Logarithmic stable determination of time dependent scalar potential from boundary measurements in a periodic quantum waveguide

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Abstract. In collaboration with *Yavar Kian* and *Eric Soccorsi*, we established a logarithmic stability estimate for the inverse problem consisting in the determination of the time dependent external potential from boundary measurements in a periodic quantum waveguide. Firstly, we reduce the original inverse problem, via a partial Floquet-Bloch-Gel'fand transform, to an inverse dynamical Schrödinger problem in a bounded domain with quasi-periodic boundary conditions in one direction. Next, following an idea by Hähner, we construct a family of complex geometric optic solutions. This family of particular solutions plays the key role in the proof the stability estimate.

A general approach to identification problems

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Abstract. In this paper a general identification problem for evolution inclusions related to linear relation is described.
Various applications to inverse problems for degenerate differential equations of weak parabolic type are considered.

New results for degenerate parabolic equations

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Abstract. In the talk it will be present some null controllability results for degenerate parabolic equations with interior degeneracy via Carleman estimates.

Chaos for the Heat Equation and the Black-Scholes Equation

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Abstract. Problems involving fair prices for stock options, bonds, etc, start with stochastic analysis and in two cases lead to deterministic parabolic equations of the form

$$\frac{\partial u}{\partial t} = ax^{\alpha}\frac{\partial^2 u}{\partial x^2} + (bx + c)\frac{\partial u}{\partial x} + du \tag{3}$$

for $0 < x < \infty$, $t > 0$, $a > 0$, and $\alpha > 0$. The two main equations are the Nobel Prize winning Black-Scholes equation (for which $\alpha = 2$) and the Cox-Ingersoll-Ross equation (for which $\alpha = 1$). There is an natural correspondence between the heat equation

$$\frac{\partial v}{\partial t} = \widetilde{a}\frac{\partial^2 v}{\partial x^2} + \widetilde{b}\frac{\partial v}{\partial x} + \widetilde{c}v \tag{4}$$

for $x \in \mathbb{R}$ and $t > 0$ and the Black-Scholes equation (1) with $\alpha = 2$ and $c = 0$ for various choics of constants with a, $\widetilde{a}$ positive. We prove that the (C_0) semigroups governing these problems are intimately connected and are chaotic on certain weighted sup norm spaces.
This is joint work with H. Emamirad, J. A. Goldstein, Ph. Rogeon.

In Search of a Better Principle of Linearized Stability

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Abstract. The nonlinear generalized heat equation

$$\frac{\partial u}{\partial t} = (-1)^{m+1}\,\Delta^m u + u^p$$

for $x \in \mathbb{R}^N$, $t > 0$ has positive radial time independent solutions. At issue is the stability or instability of these solutions. The standard versions of the Principle of Linearized Stability/Instability do not apply in this case. Nevertheless, we obtain results giving new insights into the problem, and some of the results are qualitatively dimension dependent.
This is joint work with Junqiang Han.

Null controllability of the Grushin operator with singular potential

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Abstract. We first address the null controllability problem for the parabolic equation ruled by the generalized Grushin operator $A = \partial_x^2 + |x|^{2\gamma}\partial_y^2$, $\gamma > 0$, in the rectangle $\Omega = (-1, 1) \times (0, 1)$, under an additive control supported in an open subset ω of Ω. We prove that the equation is null controllable in any positive time for $\gamma < 1$ and that there is no time for which it is null controllable for $\gamma > 1$. In the transition regime $\gamma = 1$ and when ω is a strip $\omega = (a, b) \times (0, 1)$ with $0 < a, b \leq 1$, a positive minimal time is required for null controllability.
We further investigate the null controllability property of the parabolic Grushin equation with an inverse square singular potential in the domain $D := (0, 1) \times (0, 1)$. We show that null controllability holds in large time, as in the case of the Grushin operator without potential.
The approach relies on a Fourier decomposition for the solution of the equation, that allows to reduce the problem to the validity of a uniform observability inequality with respect to the Fourier frequency. Such an inequality is obtained by means of suitable Carleman estimates, with adapted spatial weight functions.

[1] Beauchard, Karine and Cannarsa, Piermarco and Guglielmi, Roberto, *Null controllability of Grushin-type operators in dimension two*, accepted for publication in Journal of European Mathematical Society.
[2] Cannarsa, Piermarco and Guglielmi, Roberto, *Null controllability in large time for the parabolic Grushin operator with singular potential*, submitted.

Identification of a convolution kernel in a control problem for the heat equation with a boundary memory term

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Abstract. We consider a parabolic problem of the form

$$\begin{cases} D_t u(t,x) = Au(t,x) + (h * Au)(t,x) + f(t,x), & (t,x) \in (0,T) \times \Omega, \\ Bu(t,x) + (h * Bu)(t,x) + q(t,x) = M(u)(t,x) - u(t,x), & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x). \end{cases}$$

Here A is a linear strongly elliptic operator of the second order in the open subset Ω, B is a linear operator of the first order, h is a convolution kernel depending only on t, f, q and u_0 are known data. M is a nonlinear memory operator, that is, an operator that, at time t, depends only on the restriction of u to $(0,t) \times \Omega$. We suppose that h is unknown, together with u and want to reconstruct them. To this aim, we assume that, for every t the datum

$$\Phi(u(t)) = \int_\Omega \omega(x) u(t,x) dx$$

is given. In applications, a problem of this type is a model of an automatic control problem, based on a feedback device. We may also think of h as regulator that we have at disposal, in order to obtain the prescribed $\Phi(u(t))$. Particular cases which can be treated are that M is a regularization of Preisach operators or generalized plays.
The problem was studied in collaboration with C. Cavaterra (Milan).

Nematic Liquid Crystals: an Approach by Quasilinear Evolution Equation

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Abstract. Consider the (simplified) Leslie-Erickson model for the flow of liquid crystals in a bounded domain $\Omega \subset \mathbb{R}^n$. In this talk we describe a dynamic theory for these equations by analyzing it as quasilinear evolution equation. In particular, we show the existence of a unique, global, strong solutions to this system provided the initial data are close to an equilibrium. Moreover, the solution is shown to be real analytic, jointly in time and space.
This is joint work with M. Nesensohn, J. Prüss and K. Schade.

Carleman type estimates for some anisotropic systems of elasticity theory and their applications

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Abstract. We review recent results on Carleman estimates for dynamical systems of elasticity with transversal anisotropy and residual stress.
Carleman estimates are obtained by using reduction to a principal triangular system and use of second large parameter in the estimates for second order (scalar) partial differential operators. We give applications to the uniqueness and stability of the continuation and to identification of elastic parameters.

On Compactness Estimates for Hamilton-Jacobi Equations

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Abstract. Consider a first-order Hamilton-Jacobi equation

$$u_t(t,x) + H(\nabla u(t,x)) = 0, \qquad x \in \mathbb{R}^N, \quad t > 0\,, \tag{5}$$

with a strictly convex and coercive Hamiltonian $H : \mathbb{R}^N \to \mathbb{R}$. For every $\overline{u} \in W^{1,1}(\mathbb{R}^N, \mathbb{R})$, let $S_t\overline{u} \doteq u(t,\cdot)$ denote the unique viscosity solution of (5) with initial data $u(0,\cdot) = \overline{u}$. Having in mind the analysis recently developed for solutions to conservation laws [2-4], inspired by a question posed by Lax, we are interested in studying the compactifying effect of the operator S_t, at any fixed time $t > 0$, w.r.t the $W^{1,1}$-topology. Namely, we wish to estimate the Kolmogorov ε-entropy in $W^{1,1}$ of the image of bounded sets of initial data through the map S_t. We recall that, given a metric space (X, d), and a totally bounded subset K of X, we let $N_\varepsilon(K \mid X)$ denote the minimal number of sets in a cover of K by subsets of X having diameter $\leq 2\varepsilon$, and define the Kolmogorov ε-entropy of K as $H_\varepsilon(K \mid X) \doteq \log_2 N_\varepsilon(K \mid X)$. Entropy numbers play a central roles in various areas of information theory and statistics as well as of ergodic and learning theory. In the present setting, as suggested by Lax, this concept could provide a measure of the order of "resolution" of a numerical method for (5).
Our main result in [1] shows that, for every fixed $L, M > 0$, letting $C_{[L,M]}$ denote the set of Lipschitz functions $u : \mathbb{R}^N \to \mathbb{R}$ with Lipschitz constant L and with support contained in $[-M, M]^N$, there holds

$$H_\varepsilon\big(S_T(C_{[L,M]}) \mid W^{1,1}(\mathbb{R}^N, \mathbb{R})\big) \approx (1/\varepsilon^N). \tag{6}$$

Relying on fine properties of monotone operators we derive upper estimates on the ε-entropy of classes of semiconcave functions, which in turn yield upper estimates on $H_\varepsilon(S_T(C_{[L,M]}))$. Instead, lower bounds on $H_\varepsilon(S_T(C_{[L,M]}))$ are established in two steps. We first introduce a class of semiconcave functions $\mathcal{SF}$ defined as combinations of suitable bump functions, and with a combinatorial argument we provide an optimal lower estimate on the ε-entropy of such a class. Next, we prove a controllability result showing that any element of $\mathcal{SF}$ can be obtained, at any given time $T > 0$, as the value $u(t,\cdot)$ of a viscosity solution of (5), with initial data in $C_{[L,M]}$. (*Joint work with Fabio Ancona and Piermarco Cannarsa*).

[1] Ancona F., Cannarsa P., Nguyen K.T., Quantitative compactness estimates for Hamilton-Jacobi Equations, preprint (2013).
[2] Ancona F., Glass O., Nguyen K.T., Lower compactness estimates for scalar balance laws, Comm. Pure Appl. Math. 65 (2012), no. 9, 1303-1329.
[3] Ancona F., Glass O., Nguyen K.T., On compactness estimates for hyperbolic systems of conservation laws, preprint (2013).
[4] De Lellis C., Golse F., A Quantitative Compactness Estimate for Scalar Conservation Laws, Comm. Pure Appl. Math. 58 (2005), no. 7, 989–998.

Inverse problem of the variational calculus with deviating arguments

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Abstract. We consider an equation of the second order with deviating arguments and with some conditions for solution. We study the inverse problem of the variational calculus for this

equation. Namely, we seek for an integral functional for which Euler's equation coincides with given equation. We obtain conditions ensuring the solvability of the inverse problem. Moreover, we present an explicit formula for the functional, solving the inverse problem [1].
The results are demonstrated by illustrative examples.
We note that inverse problems of the variational calculus for various equations with deviating arguments have been studied, for instance, in [2]. Our approach is different from methods used by authors of this paper.

[1] V.G. Zadorozhnii, G.A. Kurina, Inverse problem of the variational calculus for differential equations of second order with deviating argument, Math. Notes, 90 (2011), no.2, 218–226.
[2] I.A. Kolesnikova, A.M. Popov, V.M. Savchin, On variational formulation for functional differential equations, J. Function Spaces and Appl., 5 (2007) , no.1, 89–101.

Gevrey class regularity for a strongly damped wave equation with hyperbolic dynamic boundary conditions

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Abstract. We consider a linear system of PDEs of the form

$$\begin{aligned} u_{tt} - c\Delta u_t - \Delta u = 0 \ in\ \Omega \times (0,T) \\ u_{tt} + \partial_n(u + cu_t) - \Delta_\Gamma(c\alpha u_t + u) = 0 \ on\ \Gamma_1 \times (0,T) \\ u = 0 \ on\ \Gamma_0 \times (0,T) \end{aligned} \tag{7}$$

on a bounded domain Ω with boundary $\Gamma = \Gamma_1 \cup \Gamma_0$. We show that the system generates a strongly continuous semigroup $T(t)$ which is analytic for $\alpha > 0$ and of Gevrey class for $\alpha = 0$. In both cases the flow exhibits a regularizing effect on the data. In particular, we prove quantitative time-smoothing estimates of the form $\|(d/dt)T(t)\| \leq |t|^{-1}$ for $\alpha > 0$, $\|(d/dt)T(t)\| \leq |t|^{-2}$ for $\alpha = 0$. The argument is based on microlocalization in the boundary collar. Moreover, when $\alpha = 0$ we prove a novel result which shows that these estimates hold under relatively bounded perturbations up to $1/2$ power of the generator.
This is a joint work with Philip J. Graber from INRIA.

Unique continuation and inverse problems for Navier-Stokes type equations

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Abstract. We investigate some inverse problems in fluid dynamic models and establish stability results via Carleman type inequalities.

Strongly ill-posed problems for regular and singular evolution equations

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Abstract The talk will be concerned with a few ill-posed problems related to *regular or singular* parabolic differential or integro-differential equations with no initial condition, but endowed with a *global* Dirichlet condition, i.e. on the whole of the lateral boundary $(0,T) \times \partial\Omega$, and a *local* differential or integro-differential Neumann condition, i.e. on $(0,T) \times \Gamma$, Γ being *an open* subset of $\partial\Omega$.

The basic questions to be answered will be uniqueness of the solution and its continuous dependence in usual Sobolev spaces. The fundamental tool will consist of Carleman estimates.
For such problems some identification problems will be dealt with, concerning uniqueness of the solution and its continuous dependence. More exactly, we shall be concerned with recovering functions depending on time.

On improvement of summability properties in nonautonomous Kolmogorov equations

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Abstract. Differently from the classical case of bounded coefficients, in general evolution operators $G(t,s)$ associated with nonautonomous elliptic operators with unbounded coefficients $A(t)$ do not improve the summability of the datum f (taken in some L^p space related to an evolution systems of measures) unless some additional assumptions on the coefficients of A(t) are prescribed. This is due to the fact that in general Sobolev embedding theorems fail to hold in the Sobolev spaces associated with evolution systems of measures. In this talk we provide some characterization of hypercontractivity, supercontractivity, ultraboundedness and ultracontractivity of the evolution operator $G(t,s)$ associated to a class of nonautonomous second order parabolic equations with unbounded coefficients defined in $I \times \mathbb{R}^d$, where I is a right-halfline. We also discuss some sufficient conditions for hypercontractivity, supercontractivity, ultraboundedness and ultracontractivity hold

On the observability of square membranes and plates

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Abstract. We discuss the observability of square membranes and plates by the Fourier method. We focus on the analysis of the observability set.
The talk is based on a joint work with Vilmos Komornik.

A variational principle for weakly coercive nonlinear diffusion equations

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Abstract. We are concerned with the study of the well-posedness of a nonlinear diffusion equation with a monotonically increasing multivalued time-dependent nonlinearity derived from a convex continuous potential having a superlinear growth to infinity. The results in this paper state that the solution of the nonlinear equation can be retrieved as the null minimizer of an appropriate minimization problem for a convex functional involving the potential and its conjugate. This approach provides new existence results under minimal growth and coercivity conditions.

Controllability of degenerate parabolic equations

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Abstract. Degenerate parabolic operators have received increasing attention in recent years because they are associated with both important theoretical analysis, such as stochastic diffusion processes, and interesting applications to engineering, physics, biology, and economics.
We studied controllability and inverse problem properties for a class of parabolic operators which may degenerate at the boundary of the space domain, in the normal direction to the boundary. Such a kind of degeneracy is relevant to study the invariance of a domain with respect to a given stochastic diffusion flow, and appears naturally in climatology models.
The aim of this talk is to explain the main results and techniques in this topic. Keywords are: global Carleman estimates, Hardy type inequalities, Bessel functions.
This is a joint work with Piermarco Cannarsa (Univ. Tor Vergata, Roma) and Judith Vancostenoble (Univ. Toulouse 3).

References: P. Cannarsa, P. Martinez, J. Vancostenoble: *Global Carleman estimates for degenerate parabolic operators with applications*, submitted.

Gelfand–Shilov Systems with White Noise Perturbations. New Trends of the Theory

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Abstract. The problem under consideration is the stochastic Cauchy problem $X'(t) = AX(t) + F(t, X) + BW(t),\ t \geq 0,\ X(0) = \xi,\ B \in \mathcal{L}(U, H)$, in Hilbert spaces H, U with operator A generating a regularized semigroup in H and an U-valued white noise perturbation W.
The problem is ill-posed through an irregular behavior of white noise process $\{W(t), t \geq 0\}$ and properties of semigroup operators generated by A: in the case of regularized semigroups solution operators to the corresponding homogeneous problem can be discontinuous and be defined not on the whole space H [1–2].
The important part of the research is development of two approaches to infinite-dimensional stochastic problems, in particular for Gelfand–Shilov systems with $A = A(i\frac{\partial}{\partial x})$. The first, we construct solutions to the problem extending the Ito (integral) form to infinite-dimensional equations. The second, we construct generalized solutions to linear problems ($F(t, X) = 0$) and develop ideas of Colombeau algebras to the nonlinear case.
The special points of the conference talk are:
• to show that both types of solutions constructed to the problem, integral and generalized, are regularized in a broad sense ;
• to demonstrate construction of solutions to Gelfand–Shilov systems with operators $A(i\frac{\partial}{\partial x})$ generating the important special case of regularized semigroups – R-semigroups.

[1] Irina V. Melnikova and Alexey Filinkov *The Cauchy problem. Three approaches*, Monographs and Surveys in Pure and Applied Mathematics, **120**, Chapman & Hall/**CRC**, 2001.
[2] Alshanskiy M.A., Melnikova I.V. *Regularized and generalized solutions of infinite dimensional stochastic problems*, Matem. sbornik, 2011, no. 11, 3–30.

Unique continuation and continuous dependence for a severely ill-posed integro-differential parabolic problem

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Abstract. The results presented here are just devoted to shed some light on the less studied subject concerning integro-differential linear equations, mainly on the questions of uniqueness

and continuous dependence on the data, two fundamental topics for people working in Applied Mathematics.
More exactly, we will deal here with an integro-differential linear parabolic problem, where the integral operators entering the equation are of Volterra type with respect to time. In our problem *no* initial condition will be supplied. It will be replaced by the requirement that the "temperature" v should have a *prescribed profile* $v(t,x) = g(t,x)$ for a.e. $(t,x) \in (0,T) \times \omega$, ω being a subdomain of the spatial domain Ω where the parabolic equation is assigned.
We will deal with integro-differential equations of form

$$D_t u(t,x) - A_0(t,x,D)\left[u(t,x) + \int_0^t k_0(t,x,s)u(s,x)\,ds\right]$$
$$+ \sum_{j=1}^{d} a_j(t,x) D_{x_j} u(t,x) + a(t,x)u(t,x) + \int_0^t k_1(t,x,s)u(s,x)\,ds$$
$$+ \int_0^t \sum_{j=1}^{d} k_{2,j}(t,x,s) D_{x_j} u(s,x)\,ds = f(t,x), \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega,$$

occur in the linear theory of heat flow in a rigid body consisting of a material with thermal memory when the heat flux is governed by

$$q(u) = c(x) \cdot \nabla_x u(t,x) + \int_0^t k_2(t,x,s) \cdot \nabla_x u(s,x)\,ds. \tag{8}$$

Consequently our equation takes into account also this physical case.
Our main task consists first in showing that the *unique continuation property* holds for our ill-posed problem and then to give an estimate regarding the continuous dependence of the solution on the data. To deduce such a result we need an additional information concerning the knowledge of the solution in the cylinder $(0,T_0) \times \Omega$, for (any) small T_0. This additional requirement is due just to the presence in our equation of the integrodifferential flux defined by (8) and highlights, once more, the troubles due to the lack of an initial condition.
In the paper [1] we dealt with the simpler equation where the internal energy

$$e(u)(t,x) = \int_0^t h(t,x,s)u(s,x)\,ds$$

was in the place of the missing heat flux with memory. Moreover, the term corresponding to $e(v)$ entering the balance equation was given by

$$D_t e(v)(t,x) = h(t,x,t)v(t,x) + \int_0^t D_t h(t,x,s)v(s,x)\,ds.$$

We proved in [1] both uniqueness and continuous dependence. *A posteriori* we can say that the case corresponding to the integral internal energy is somehow simpler than the present case corresponding to the integrodifferential flux prescription.
We stress that the quoted paper is the starting point to treat the present more complex case. However to deduce a continuous dependence result our computations seem to require some additional information on the solution for small time.
Work in co-operation with Alfredo Lorenzi.

[1] A. Lorenzi, F. Messina: *Unique continuation and continuous dependence results for a severely ill-posed integro-differential parabolic problem*, J. Inverse Ill-Posed Problems 20 (2012), pp 615-636.

Elliptic operators with unbounded diffusion coefficients in L^p spaces

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Abstract. In this talk we focus our attention on elliptic operators with unbounded diffusion of the form

$$Lu = (1 + |x|^\alpha)\Delta u, \tag{9}$$

for positive values of α, on $L^p = L^p(\mathbb{R}^N, dx)$ with respect to the Lebesgue measure. The case $\alpha \le 2$ has been already investigated by S. Fornaro and L. Lorenzi who proved that the operator above generates a strongly continuous and analytic semigroup in L^p and in spaces of continuous functions. For $1 < p < \infty$ an explicit description follows from the a-priori estimates

$$\|(1 + |x|^\alpha)D^2u\|_p \le C(\|u\|_p + \|(1 + |x|^\alpha)\Delta u\|_p.$$

Similar estimates hold for a mor general class of operators. They can be deduced by some weigthed norm inequalities for Caldéron-Zygmund singular integrals. Muckenhoupt and Wheeden for example proved that estimates of the form

$$\|aD^2u\|_p \le C\|a\Delta u\|_p$$

are true for weights a in suitable Muckenhoupt classes. In particular the estimates above imply that

$$\||x|^\alpha D^2u\|_p \le C\||x|^\alpha \Delta u\|_p \tag{10}$$

and

$$\|(1 + |x|^\alpha)D^2u\|_p \le C(\|u\|_p + \|(1 + |x|^\alpha)\Delta u\|_p$$

for $0 < \alpha < \frac{N}{p'}$ where p' is the conjugate exponent of p.
Similar estimates follow also by the work of Kree who studied singular integrals in L^p spaces with respect to the weight $1 + |x|^\alpha$, $-\frac{N}{p'} < \alpha < \frac{N}{p'}$.
We will prove that for $2 < \alpha \le (N - 2)(p - 1)$ and $N \ge 3$ the operator above generates a semigroups in L^p which is analytic when for $\alpha < (N - 2)(p - 1)$. Moreover for $2 < \alpha < N/p'$ an explicit description of the domain follows from an improved version of the a-priori estimates (10).
Joint work with C. Spina.

A semigroup approach to the generalized Cox-Ingersoll-Ross problem

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Abstract. The topic of my talk concerns some of the results obtained in a joint work [1] with Gisele Ruiz Goldstein, Jerome A. Goldstein (University of Memphis) and Silvia Romanelli (University of Bari).
We considered the following initial value parabolic problem

$$(CIR)_r \quad \begin{cases} \dfrac{\partial u}{\partial t} = \nu^2 x \dfrac{\partial^2 u}{\partial x^2} + \gamma \dfrac{\partial u}{\partial x} + \beta x \dfrac{\partial u}{\partial x} - r x u, & t \ge 0,\ x \ge 0, \\ u(0, x) = f(x), & x \ge 0 \end{cases} \tag{11}$$

with $f \in X = C[0, \infty]$, where $\nu > 0$, $\gamma > 0$, $\beta \in \mathbb{R}\backslash\{0\}$ and $r \ge 0$ are unknown parameters.

When $r = 1$, the problem (11) is known in Financial Mathematics, in the framework of the theory of short-term interest rate models, as the Cox-Ingersoll-Ross (CIR) problem (see [2]) for discounted asset prices, which depend on time t and the current value x of only one random variable, the spot interest rate $Y(t)$, whose dynamics is described by the following Stochastic Differential Equation:

$$dY(t) = (\gamma + \beta Y(t))\, dt + \sqrt{2}\, \nu \sqrt{Y(t)}\, dW(t), \quad Y(0) = x_0. \tag{12}$$

$\{W_t, t \geq 0\}$ is a standard one-dimensional Wiener process.
We studied the semigroup of $(\mathrm{CIR})_r$ problem on spaces of continuous functions on $(0, \infty)$, which have finite limits at 0 and ∞. We derive a semi explicit representation of the semigroup and a Feynman-Kac type formula, in a generalized sense, for the unique solution of the $(\mathrm{CIR})_1$ initial value problem as a useful tool for understanding additional properties of the solution itself.

[1] G. Ruiz Goldstein, J.A. Goldstein, R.M. Mininni and S. Romanelli, The semigroup governing the generalized Cox-Ingersoll-Ross equation, *Preprint* (2012). Submitted.
[2] J.C. Cox, J.E. Ingersoll, S.A. Ross, A theory of the term structure of interest rates, Econometrica, 53(1985), pp. 385–407.

The Penrose-Fife phase-field system with dynamic boundary conditions

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Abstract. Our aim in this talk is to discuss the well-posedness and longtime behavior of the Penrose-Fife system in phase transition with dynamic boundary conditions.

Recovering the time-dependent reaction coefficient in a linear parabolic equation

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Abstract. Let H be a real separable Hilbert space and $A : \mathcal{D}(A) \to H$ be a positive and self-adjoint (unbounded) operator, and consider the abstract parabolic Cauchy problem

$$u'(t) + A\, u(t) = \mu(t)\, u(t), \quad t \in (0, T), \quad u(0) = \phi \in H. \tag{13}$$

We investigate the inverse problem of recovering, along with the H-valued function u, also the time-dependent *reaction coefficient* $\mu(t)$, by imposing the *overdeterminating condition* expressed by the further measurement of $u(t)$ in the H-norm $\|\cdot\|$; that is

$$\|u(t)\|^2 = \rho(t), \quad t \in (0, T). \tag{14}$$

Here $\rho(t)$ is a known function, supposed to be strictly positive on $[0, T]$.

We prove the well-posedness of problem (13)-(14) in the sense of Hadamard, i.e. existence and uniqueness of a couple (u, μ), as well as its continuous dependence on the given data ϕ and ρ. This is done by adapting to the inverse problem a finite-dimensional approximation scheme which allows to use suitable representation formulas. Such an approximation scheme is proved to converge to a unique solution whenever the initial value ϕ is assumed to belong to a slightly more regular space than H (namely, in an interpolation space between H and $\mathcal{D}(A)$, arbitrarily close to H). The problem in the limiting case $\phi \in H$ is still open.
Joint work with Roberto Guglielmi.

Linear evolution equation of "hyperbolic" type in Hilbert space — new observation

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Abstract. Let $\{A(t); 0 \le t \le T\}$ be a family of closed linear operators in a complex Hilbert space X. This talk is concerned with linear evolution equations of the form

$$du(t)/dt + A(t)u(t) = f(t) \quad \text{on} \quad (0,T). \tag{E}$$

Let S be a selfadjoint operator in X, satisfying $(u, Su) \ge \|u\|^2$ for $u \in D(S)$. Assume that the following five conditions are satisfied:

(**I**) There is a constant $\alpha \ge 0$ such that

$$\mathrm{Re}\,(A(t)w, w) \ge -\alpha\|w\|^2 \quad \forall\, w \in D(A(t)),\ \forall\, t \in [0,T];$$

(**II**) $Y := D(S^{1/2}) \subset D(A(t)),\ 0 \le t \le T$;

(**III**) There is a constant $\beta \ge \alpha$ such that

$$\mathrm{Re}\,(A(t)v, Sv) \ge -\beta(v, Sv) \quad \forall\, v \in D(S),\ \forall\, t \in [0,T];$$

(**IV**) (new) There is a constant $\gamma \ge \beta$ such that

$$\mathrm{Re}\,(A(t)u, S^2u) \ge -\gamma(u, S^2u) \quad \forall\, u \in D(S^2),\ \forall\, t \in [0,T];$$

(**V**) $A(\cdot) \in C_*\big([0,T]; B(Y,X)\big)$, that is, $A(\cdot)$ is strongly continuous on $[0,T]$ to $B(Y,X)$.

Then one has

Theorem. Let $f(\cdot) \in C([0,T];X) \cap L^1(0,T;Y)$. Assume that conditions (**I**) – (**V**) are satisfied. Then for any $u_0 \in Y$ there exists a unique solution $u(\cdot)$ to (E) with $u(0) = u_0$ such that

$$u(\cdot) \in C^1([0,T];X) \cap C([0,T];Y).$$

Remark Previously, instead of (**IV**) we had imposed conditions (**I**) and (**III**) with $\mathrm{Re}\,(A(t)w, w)$ and $\mathrm{Re}\,(A(t)v, Sv)$ replaced with $\mathrm{Re}\,(\pm A(t)w, w)$ and $\mathrm{Re}\,(\pm A(t)v, Sv)$, respectively:

$$|\mathrm{Re}\,(A(t)w, w)| \le \alpha\|w\|^2, \quad |\mathrm{Re}\,(A(t)v, Sv)| \le \beta(v, Sv).$$

Therefore the result was not applied to parabolic problems.

Heat Semigroup and geometric measure theory in Carnot groups

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Abstract. Two different characterizations of sets with finite perimeter and functions of bounded variation in Carnot groups are presented, in terms of the short-time behaviour of the heat semigroup.

[1] M. Bramanti, M. Miranda, D. Pallara: Two characterization of BV functions on Carnot groups via the heat semigroup, *Int. Math. Res. Not.,* **17** (2012), 3845-3876.

Controllability of systems with memory:
a survey of recent results

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Abstract. In recent years controllability properties of heat equations with memory have been studied using different methods. The case that attracted the main attention is

$$\theta_t = \int_0^t N(t-s)\Delta\theta(s)\,\mathrm{d}s$$

under smoothness properties of the kernel $N(t)$, when $N(0) > 0$. In fact, additional restrictions on the kernel (imposed by thermodynamics) do not have a role in the study of controllability; even the sharp control time.
This equation is a perturbed wave equation and it has been proved that it shares the controllability properties of the wave equation. Furthermore, the memory allows for more degrees of freedom and recent results show independence/controllability of the pair (flux/temperature).
A model which looks more similar to the standard heat equation is

$$\theta_t = \Delta\theta + \int_0^t N(t-s)\Delta\theta(s)\,\mathrm{d}s\,.$$

Controllability of this equation has been rarely studied but striking negative results have been reported recently, on special examples: for every time T there exist initial conditions which cannot be controlled to hit the target 0.
In this talk we shall discuss these results, and recent extensions, obtained using moment methods.

Stabilization of second-order
evolution equations with time delay

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Abstract. We consider secondorder evolution equations with damping and time delay feedback and give an exponential stability result under appropriate conditions. Some applications of our abstract framework to concrete models (wave equation, elasticity system, Petrovsky system) are also discussed.

Null controllability of viscous
Hamilton-Jacobi equations

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Abstract. We study the problem of null controllability for viscous Hamilton-Jacobi equations with controls localized in the interior of a bounded domain. We prove that any initial datum can be driven to zero in sufficiently large time, but a waiting time may actually occurr, which we estimate in terms of sup-norm of initial data. Exact control to trajectories is discussed similarly. Proofs combine controllability of linearized operator and large time decay of solutions, as well as barrier estimates for the waiting time. Joint work with E. Zuazua.

Well posedness of the stabilization of the Navier-Stokes equations
with mixed boundary conditions and partial informations

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Abstract. We shall address the problem of stabilizing the incompressible Navier-Stokes equations about an unstable stationary solution by boundary controls, in the case of partial information coming from boundary measurements, and with mixed boundary conditions. The presence of mixed boundary conditions in domains with corners makes the analysis of the Navier-Stokes equations trickier. The boundary measurements, typically pressure measurements, are also a new source of difficulty when we want to estimate the velocity fields. We prove that we can couple an estimator and a feedback control law in order to stabilize locally the full nonlinear system.

A weighted Hardy inequality and nonexistence of positive solutions to some nonlinear problems

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Abstract. In this talk, we prove that the following weighted Hardy inequality

$$\left(\frac{|d-p|}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p}\,d\mu \le \int_\Omega |\nabla u|^p\,d\mu + \left(\frac{|d-p|}{p}\right)^{p-1} \operatorname{sgn}(d-p) \int_\Omega |u|^p \frac{(x^tAx)^{p/2}}{|x|^p}\,d\mu \quad (15)$$

holds with optimal Hardy constant $\left(\frac{|d-p|}{p}\right)^p$ for all $u \in W^{1,p}_{\mu,0}(\Omega)$ if the dimension $d \ge 2$, $1 < p < d$, and for all $u \in W^{1,p}_{\mu,0}(\Omega \setminus \{0\})$ if $p > d \ge 1$. Here we assume that Ω is an open subset of $\mathbb{R}^d$ with $0 \in \Omega$, A is a real $d \times d$-symmetric positive definite matrix, $c > 0$, and

$$d\mu := \rho(x)\,dx \qquad \text{with density} \qquad \rho(x) = c \cdot \exp(-\frac{1}{p}(x^tAx)^{p/2}), \quad x \in \Omega.$$

Due to the optimality of the Hardy constant in (15) and by using the Cabré-Martel approach [2], we can establish nonexistence (locally in time) of positive weak solutions of a p-Kolmogorov parabolic equation perturbed by a singular potential. This generalizes the pioneering work [1] and more recently [3].

[1] P. Baras, J.A. Goldstein: *The heat equation with singular potential*, Trans. Amer. Math. Soc. **284** (1984), 121-139.
[2] X. Cabré, Y. Martel: *Existence versus explosion instantanée pour des équations de la chaleur linéaires avec potentiel singulier*, C.R. Acad. Sci. Paris **329** (1999), 973-978.
[3] G.R. Goldstein, J.A. Goldstein, A. Rhandi: *Weighted Hardy's inequality and the Kolmogorov equation perturbed by an inverse-square potential*, Applicable Analysis **91** (2012), 2057-2071.

Analytic semigroups generated by differential operators with interior degeneracy

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Abstract. We consider operators of the type $A_1u = (au')'$, or $A_2u = au''$, where $a \in C[0,1]$ vanishes in $x_0 \in (0,1)$ and $a(x) > 0$ for any $x \in [0,1]\setminus\{x_0\}$. According to the order of degeneracy and additional assumptions, we state existence and analyticity of the generated semigroups in suitable Hilbert spaces.

Phase-field methods for direct and inverse problems

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Abstract. We shall show how phase-field functions may be used to approximate the boundaries of sets modeling defects or obstacles characterized by homogeneous Neumann conditions. Applications include, for example, the inverse crack problem for insulating cracks or the cloaking problem for the Helmholtz equation.

Reachability results for integral equations and the coupling problem

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Abstract. The purpose of the talk is to provide a survey of some control problems for evolution equations with nonlocal terms of convolution type. In particular, we focus on observability estimates obtained by means of non-harmonic analysis techniques. A coupling problem is also exhibited. Joint works with Paola Loreti.

L^p-theory for second-order elliptic operators via new decomposition of unbounded drift terms

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Abstract. In this talk we consider second-order elliptic operators with **unbounded** coefficients of the form $Au := -\mathrm{div}(a\nabla u) + F \cdot \nabla u + Vu$ in $L^p(\mathbb{R}^N)$ ($N \in \mathbb{N}$, $1 < p < \infty$). We present a natural condition for the m-accretivity and m-sectoriality of A in the sense of Goldstein's book by introducing the identity for a decomposition of unbounded drift terms:

$$\begin{aligned}\int_{\mathbb{R}^N} (Au)\overline{v}\,dx \ &= \int_{\mathbb{R}^N} (a\nabla u \cdot \nabla \overline{v})\,dx + \int_{\mathbb{R}^N} \Big(V - \frac{\mathrm{div}F}{p}\Big) u\overline{v}\,dx \\ &\quad + \int_{\mathbb{R}^N} F \cdot \Big(\frac{\overline{v}\nabla u}{p'} - \frac{u\nabla \overline{v}}{p}\Big)\,dx.\end{aligned}$$

As an example, we deal with an L^p-generalization of Kato's example for selfadjointness described in his paper in 1981.

Homogeneous Calderón-Zygmund estimates for a class of second order elliptic operators

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Abstract. Given a uniformly elliptic operator $L = \sum_{i,j=1}^N a_{ij}(x)D_{ij}$, with (a_{ij}) bounded and uniformly continuous (BUC) functions in $\mathbb{R}^N$, $N \geq 2$, a-priori estimates and solvability results in Sobolev spaces for the associated Poisson problem are well known in literature. In this context, a basic role is played by the classical inequality

$$\|D^2 u\|_p \leq C(\|Lu\|_p + \|u\|_p), \quad u \in W^{2,p},$$

that leads, in addition, to the unique resolution of the resolvent equation $Lu - \lambda u = f$, $\lambda > 0$. We are interested, among other things, in establishing the *stronger* homogeneous estimate

$$\|D^2 u\|_p \leq C\|Lu\|_p\,, \quad D^2 u \in L^p\,. \tag{16}$$

To the best of our knowledge, results concerning the validity of (16) have been proved only in certain special cases.

We show that, under the assumptions that the $a_{ij}(x)$ are strongly elliptic, BUC and possess a limit as $|x| \to \infty$, for any given $f \in L^p$ equation $Lu = f$ has one and only one solution in *homogeneous Sobolev spaces* satisfying (16). On the other hand, we also exhibit an example which shows that if the condition of the existence of the limit is removed, then inequality (16) is not true. Thus, this condition is clearly pivotal for the validity of our result.
As a corollary to the above result, we are able to show the resolvent estimate

$$\|(\lambda - L)^{-1} f\|_p \leq \frac{C}{\lambda} \|f\|_p,$$

for any $\lambda > 0$, and with $C = C(p) > 0$.
Joint work with G.P. Galdi, G. Metafune, C. Tacelli.

An inverse problem for a third order PDE arising in high-intensity ultrasound: global uniqueness and stability by one boundary measurement

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Abstract. Both canonical recovery (inverse) problems of (i) uniqueness and (ii) stability are investigated for a third order (in time) PDE arising in high-intensity ultrasound, by means of just one boundary measurement performed on an appropriate portion of the boundary. Carleman estimates in Lasiecka-Triggiani-Zhang's work (2000) for second order hyperbolic equations are one of the key tools of the investigation.
This is joint work with Shitao Liu.

Source Identification in a Semilinear Evolution Equation with Delay

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Abstract. An existence, uniqueness and continuous dependence on the data result for a source term identification problem in a semilinear functional delay differential equation in a general Banach space is established. As additional condition, it is assumed that the mean of the solution, with respect to a non-atomic Borel measure, is a preassigned element in the domain of the linear part of the right-hand side of the equation. Two applications to source identification, one in a parabolic functional delay equation and another one in a hyperbolic delay equation, are also discussed.
Joint work with Alfredo Lorenzi, Università degli Studi di Milano.

Isomorphism, discreteness of the spectrum and completeness of root vectors for regular BVPs for elliptic DOEs of the fourth order depending on a parameter

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Abstract. We treat some fourth order elliptic differential-operator boundary value problems on a finite interval quadratically depending on a parameter. We prove an isomorphism result (which implies maximal L_p-regularity) in the corresponding abstract Sobolev spaces. The underlying space is a UMD Banach space. Then, for the corresponding homogeneous problems, we prove discreteness of the spectrum and two-fold completeness of a system of eigenvectors and associated vectors of the problem in the framework of Hilbert and UMD Banach spaces. We apply the obtained abstract results to non-local boundary value problems for elliptic and quasi-elliptic equations with a parameter in (bounded and unbounded) cylindrical domains.